

# STOCHASTIC FIELD THEORY FOR TRANSPORT STATISTICS IN DIFFUSIVE SYSTEMS

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We present a field theory for the statistics of charge and current fluctuations in diffusive systems. The cumulant generating function is given by the saddle-point solution for the action of this field theory. The action depends on two parameters only: the local diffusion and noise coefficients, which naturally leads to the universality of the transport statistics for a wide class of multi-dimensional diffusive models. Our theory can be applied to semi-classical mesoscopic systems, as well as beyond mesoscopic physics.

## 1 Introduction

In this short paper, we present the essential part of our theory<sup>1</sup> for the statistics of current and density fluctuations in non-equilibrium diffusive systems. The theory is based on the stochastic path integral approach to the statistics of fluctuations in networks,<sup>2</sup> and represents an alternative to quantum methods<sup>3,4</sup> for the evaluation of the full counting statistics (FCS) of the transmitted charge, as well as to classical cascade correction method<sup>5</sup> for the evaluation of current cumulants.

The building blocks of our theory are a separation of the time scales (sources of noise are Markovian) and a saddle-point approximation of the resulting functional integral. The separation of variables into fast currents and slow varying generalized charges suggests the network description of a system introduced in Sec. 2. In Sec. 3 we consider a large number of nodes and derive the continuum limit for the stochastic path integral, which turns out to be a stochastic field theory with an action being a function of only two quantities, the diffusion  $D$  and noise  $F$  functionals of the charge density  $\rho$ . This naturally leads to the universality of the FCS not only for the mesoscopic diffusive conductor [see Eq. (22)], but for a whole class of multi-dimensional diffusive models (see also Ref. [1] for details).

We would like to stress that due to its essentially classical construction, our stochastic field theory is not limited to mesoscopic physics,<sup>6</sup> but can also be applied to reaction-diffusion systems,<sup>7</sup> symmetric exclusion processes,<sup>8</sup> and many other areas of classical stochastic processes.<sup>9</sup>

## 2 Stochastic Path Integral for a Network

Consider a network with the state of each node  $\alpha$  described by one charge  $Q_\alpha$  ( $\mathbf{Q}$  is the charge vector describing the charge state of the network). The node's state may be changed by transport: flow of charges between nodes takes place via the connectors carrying currents  $I_{\alpha\beta}$  from node  $\alpha$  to node  $\beta$ . The rate of change of these charges  $Q_\alpha$  is given by

$$\dot{Q}_\alpha = \sum_\beta I_{\alpha\beta}, \quad P_{\alpha\beta}(I_{\alpha\beta}) = \int \frac{d\lambda_{\alpha\beta}}{2\pi} \exp\{-itI_{\alpha\beta}\lambda_{\alpha\beta} + tH_{\alpha\beta}(\lambda_{\alpha\beta})\}, \quad (1)$$

where  $P_{\alpha\beta}$  is the probability distribution of the fast current between nodes  $\alpha$  and  $\beta$ , and  $H_{\alpha\beta}$  is the current cumulant generating function. The fact that  $P_{\alpha\beta}$  also depends on the charges  $\mathbf{Q}$  is one source of the difficulty of the problem.

Assuming that the generators  $H_{\alpha\beta}$  of the fluctuating currents  $I_{\alpha\beta}$  are known, we seek an evolution of the probability distribution  $\Gamma(\mathbf{Q}, t)$  of the set of charges  $\mathbf{Q}$  for a given initial condition  $\Gamma(\mathbf{Q}, 0)$ . In other words, one has to find the conditional probability (which we refer to as the evolution operator)  $U(\mathbf{Q}, \mathbf{Q}', t)$  such that

$$\Gamma(\mathbf{Q}, t) = \int d\mathbf{Q}' U(\mathbf{Q}, \mathbf{Q}', t) \Gamma(\mathbf{Q}', 0). \quad (2)$$

We assume that there is a separation of time scales,  $\tau_0 \ll \tau_C$ , between the correlation time of current fluctuations,  $\tau_0$ , and the slow relaxation time of charges in the nodes,  $\tau_C$ . In Ref. [2] we have used the separation of time scales to derive a stochastic path integral representation for the evolution operator,

$$U(\mathbf{Q}_f, \mathbf{Q}_i, t) = \int \mathcal{D}\mathbf{Q} \mathcal{D}\mathbf{\Lambda} \exp\{S(\mathbf{Q}, \mathbf{\Lambda})\}, \quad (3)$$

$$S(\mathbf{Q}, \mathbf{\Lambda}) = \int_0^t dt' [-i\mathbf{\Lambda} \cdot \dot{\mathbf{Q}} + (1/2) \sum_{\alpha\beta} H_{\alpha\beta}(\mathbf{Q}, \lambda_\alpha - \lambda_\beta)]. \quad (4)$$

The variables  $\lambda_\alpha$  are auxiliary variables for every node that impose charge conservation in the network.

### 3 Continuum Limit

From the stochastic network, it is straightforward to go to spatially continuous systems as the spacing between the nodes is taken to zero. Consider a series of identical, equidistant nodes separated by a distance  $\Delta z$ . This nodal chain could represent a chain of chaotic cavities, Fig. 1, in a mesoscopic context.<sup>10,11</sup> The sum over  $\alpha$  and  $\beta$  becomes a sum over each node in space connected to its neighbors. The action for this arrangement is

$$S = \int_0^t dt' \sum_{\alpha} \{-\lambda_{\alpha} \dot{Q}_{\alpha} + H(Q_{\alpha}, Q_{\alpha-1}; \lambda_{\alpha} - \lambda_{\alpha-1})\}, \quad (5)$$

where for simplicity we have chosen real counting variables,  $i\lambda_{\alpha} \rightarrow \lambda_{\alpha}$ . The only constraint made on  $H$  is that probability is conserved,  $H(\lambda_{\alpha} - \lambda_{\alpha-1}) = 0$  for  $\lambda_{\alpha} = \lambda_{\alpha-1}$ . We now derive a lattice field theory by formally expanding  $H$  in  $\lambda_{\alpha} - \lambda_{\alpha-1}$  and  $Q_{\alpha} - Q_{\alpha-1}$ . Only differences of the counting variables will appear in the series expansion, while we must keep the full  $Q$  dependence of the Hamiltonian. If there are  $N \gg 1$  nodes in the lattice, for fixed boundary conditions the difference between adjacent variables,  $\lambda_{\alpha} - \lambda_{\alpha-1}$  and  $Q_{\alpha} - Q_{\alpha-1}$  will be of order  $1/N$ , and therefore provides a good expansion parameter. The expansion of the Hamiltonian (5) to second order in the difference variables gives

$$H = \frac{\partial H}{\partial \lambda_{\alpha}} (\lambda_{\alpha} - \lambda_{\alpha-1}) + \frac{1}{2} \frac{\partial^2 H}{\partial \lambda_{\alpha}^2} (\lambda_{\alpha} - \lambda_{\alpha-1})^2 + \frac{\partial^2 H}{\partial Q_{\alpha} \partial \lambda_{\alpha}} (Q_{\alpha} - Q_{\alpha-1})(\lambda_{\alpha} - \lambda_{\alpha-1}), \quad (6)$$

where the expansion coefficients are evaluated at  $\lambda_{\alpha} = \lambda_{\alpha-1}$  and  $Q_{\alpha} = Q_{\alpha-1}$  and are functions of  $Q_{\alpha-1}$ . Terms involving only differences of  $Q_{\alpha} - Q_{\alpha-1}$  are zero because  $H(\lambda_{\alpha} - \lambda_{\alpha-1}) = 0$  for  $\lambda_{\alpha} = \lambda_{\alpha-1}$ . All terms in Eq. (6) need explanation. First, the expression  $\partial H / \partial \lambda_{\alpha}$  is the local current at zero bias (because the charges in adjacent nodes are equal) which will usually be zero. There may be exceptional circumstances where this term should be kept, but we do not consider them here. The term  $\partial^2 H / \partial Q_{\alpha} \partial \lambda_{\alpha} = -G(Q_{\alpha-1})$  is the linear response of the current to a charge difference. Hence,  $G$  is the generalized conductance<sup>12</sup> of the connector between nodes  $\alpha$  and  $\alpha - 1$ .  $\partial^2 H / \partial \lambda_{\alpha}^2 = C(Q_{\alpha-1})$  is the current noise through the same connector because  $H$  is the generator of current cumulants.

We are now in a position to take the continuum limit by replacing the node index  $\alpha$  with a coordinate  $z$ , introducing the fields  $Q(z)$ ,  $\lambda(z)$ , and making the expansions

$$\lambda_{\alpha} - \lambda_{\alpha-1} \rightarrow \lambda' \Delta z + (1/2) \lambda'' (\Delta z)^2 + \mathcal{O}(\Delta z)^3, \quad (7)$$

$$Q_{\alpha} - Q_{\alpha-1} \rightarrow Q' \Delta z + (1/2) Q'' (\Delta z)^2 + \mathcal{O}(\Delta z)^3. \quad (8)$$

The action may now be written in terms of intensive fields by scaling away  $\Delta z$ ,

$$H \rightarrow h(\rho, \lambda) \Delta z, \quad Q_{\alpha} \rightarrow \rho(z) \Delta z, \quad G_{\alpha}(\Delta z)^2 \rightarrow D(\rho), \quad C_{\alpha} \Delta z \rightarrow F(\rho), \quad (9)$$

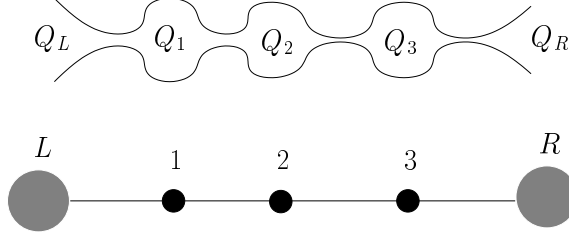


Figure 1: A one dimensional lattice of nodes connected on both ends to reservoirs. This situation could represent a series of mesoscopic chaotic cavities connected by point contacts.

and taking the limit  $\sum_{\alpha} H \rightarrow \int dz h(\rho, \lambda)$ . One may check that expanding the Hamiltonian to higher than second order in  $\Delta z$  will result in terms suppressed by powers of  $\Delta z/L$  and consequently vanish as  $\Delta z \rightarrow 0$ .

These considerations leave the one dimensional action as

$$S = - \int_0^t dt' \int_0^L dz \left[ \lambda \dot{\rho} + D \rho' \lambda' - \frac{1}{2} F (\lambda')^2 \right]. \quad (10)$$

Here  $D$  is the local diffusion constant and  $F$  is the local noise density which are discussed in detail below. It is very important that these two functionals  $D, F$  are all that is needed to calculate current statistics. Classical field equations may be obtained by taking functional derivatives of the action with respect to the charge and counting fields:

$$\dot{\lambda} = -\frac{1}{2} \frac{\delta F}{\delta \rho} (\lambda')^2 - D \lambda'', \quad \dot{\rho} = [-F \lambda' + D \rho']'. \quad (11)$$

We have to solve these coupled differential equations subject to the boundary conditions  $\rho(t, 0) = \rho_L(t)$ ,  $\rho(t, L) = \rho_R(t)$ ,  $\lambda(t, 0) = \lambda_L(t)$ , and  $\lambda(t, L) = \lambda_R(t)$ . Functions  $\lambda_L(t)$  and  $\lambda_R(t)$  are the counting variables of the absorbed charges at the left and right end of the system. Once Eqs. (11) are solved, the solutions  $\rho(z, t)$  and  $\lambda(z, t)$  should be substituted back into the action (10) and integrated over time and space. The resulting function,  $S_{sp}[\rho_L(t), \rho_R(t), \lambda_L(t), \lambda_R(t), t, L]$  is the generating function for time-dependent cumulants of the current distribution. Often, the relevant experimental quantities are the stationary cumulants. These are given by neglecting the time dependence, finding static solutions,  $\dot{\rho} = \dot{\lambda} = 0$ , and imposing static boundary conditions. We can also introduce sources  $\int dt dz \chi(z, t) \rho(z, t)$  and calculate density correlation functions.

To justify the saddle-point approximation, it is useful to define dimensionless variables. The boundary conditions  $\rho_L$ , and  $\rho_R$  provide the charge density scale  $\rho_0$  in the problem, so we define  $\rho(z) = \rho_0 f(z)$ , where  $f \sim 1$  is an occupation. We furthermore rescale  $z \rightarrow Lz$ , and  $t \rightarrow \tau_D t$ , where  $\tau_D = L^2/D$  is the diffusion time, thus obtaining

$$S = -L\rho_0 \int_0^t dt' \int_0^1 dz' \left[ \lambda \dot{f} + f' \lambda' - \frac{F}{2D\rho_0} (\lambda')^2 \right]. \quad (12)$$

We assume that the combination  $F/D\rho_0$  is of order 1. From Eq. (12), the dimensionless large parameter is  $\gamma = \rho_0 L \gg 1$ , i.e. the number of transporting charge carriers. The saddle-point contribution is of order  $\gamma t/\tau_D$ , while the fluctuation contribution is of order  $t/\tau_D$ .

Repeating these steps in multiple dimensions yields the action

$$S = - \int_0^t dt' \int_{\Omega} d\mathbf{r} \left[ \lambda \dot{\rho} + \nabla \lambda \hat{D} \nabla \rho - (1/2) \nabla \lambda \hat{F} \nabla \lambda \right], \quad (13)$$

where  $\hat{F}$  and  $\hat{D}$  are general matrix functions of the density  $\rho$  and coordinate  $\mathbf{r}$  which should be interpreted as noise and diffusion matrices.

As in any field theory, symmetries of the action play an important role because they lead to conserved quantities. We first note that the Hamiltonian  $h(\rho, \nabla\rho, \nabla\lambda)$  is a functional of  $\nabla\lambda$  alone with no  $\lambda$  dependence. This symmetry is analogous to gauge invariance, and leads to the equation of motion

$$\dot{\rho} + \nabla \cdot \mathbf{j} = 0, \quad \mathbf{j} = -\hat{D}\nabla\rho + \hat{F}\nabla\lambda, \quad (14)$$

which can be interpreted as conservation of the conditional current density  $\mathbf{j}$ . The next symmetry is related to the invariance under a shift in the space and time coordinates  $\{\delta\mathbf{r}, \delta t\}$ . This symmetry leads to equations analogous to the conservation of the local energy/momentum tensor.<sup>13</sup> For the stationary limit (where  $\dot{\rho}$  and  $\dot{\lambda}$  vanish) and for symmetric diffusion and noise tensors, the conservation law is relatively simple and is given by

$$\sum_m \nabla_m T_{mn} = 0, \quad T_{mn} = j_m(\nabla_n \lambda) - (\nabla_n \rho)(D\nabla\lambda)_m - h\delta_{mn}. \quad (15)$$

For the special case of a one dimensional geometry, the Hamiltonian itself is the conserved quantity (see Sec. 4).

#### 4 FCS of Diffusive Systems

We first consider the general 1D field theory with the action (10), and then demonstrating our solution for the FCS of the mesoscopic diffusive wire specifically. In the stationary limit,  $\dot{\rho} = \dot{\lambda} = 0$ , the action can be written as

$$S = t \int_0^L dz \left[ -D\rho'\lambda' + \frac{1}{2}F(\lambda')^2 \right]. \quad (16)$$

The stationary saddle-point equations

$$(F\lambda' - D\rho')' = 0, \quad 2D\lambda'' + \frac{\delta F}{\delta \rho}(\lambda')^2 = 0, \quad (17)$$

can be partially integrated leading to the following two equations:

$$D\rho' = \pm\sqrt{\mathcal{I}^2 - 2\mathcal{H}F}, \quad (18)$$

$$\lambda' = 2\mathcal{H}/(\mathcal{I} - D\rho'). \quad (19)$$

The two integration constants  $\mathcal{I} = -D\rho' + F\lambda'$  and  $\mathcal{H} = -D\rho'\lambda' + (F/2)(\lambda')^2$  are the conserved (conditional) current and the Hamiltonian density, respectively. These conservation laws follow from the symmetries of our 1D field theory [see Eqs. (14) and (15) and the surrounding discussion]. Thus we obtain the following result for the action (16),

$$S = tL\mathcal{H}. \quad (20)$$

The Eqs. (18-20) represent the formal solution of the FCS problem for 1D diffusion models with  $D(\rho)$  and  $F(\rho)$  being arbitrary functions of  $\rho$ . The following procedure has to be done in order to obtain the cumulant generating function  $S(\chi)$  of the transmitted charge: (i) The differential equation (18) has to be solved for  $\rho(z)$  with the boundary conditions  $\rho(z)|_{z=0} = \rho_L$  and  $\rho(z)|_{z=L} = \rho_R$ . The constant  $\mathcal{I}$  should be expressed through the constants  $\rho_L$ ,  $\rho_R$ , and  $\mathcal{H}$ . (ii) Next,  $\rho(z)$  is substituted into Eq. (19) which is integrated to obtain  $\lambda(z)$  with the boundary conditions  $\lambda_L = 0$  and  $\lambda_R = \chi$ . (iii) Finally, using the solution for  $\lambda(z)$  the constant  $\mathcal{H}$  is expressed in terms of  $\rho_L$ ,  $\rho_R$ ,  $\chi$ , and substituted into the action (20). We note that by

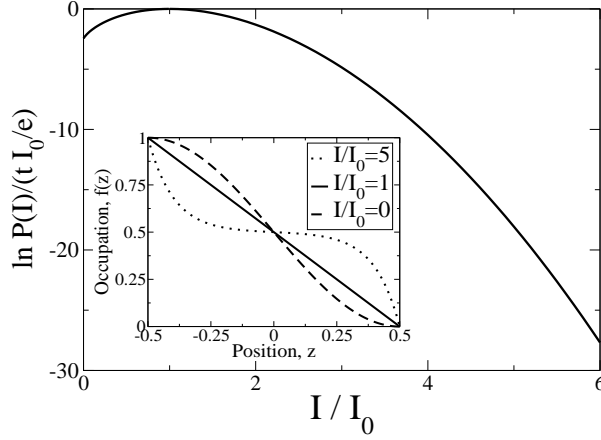


Figure 2: The logarithm of the distribution of the current through a mesoscopic diffusive wire as a function of the ratio  $I/I_0$  of the current to its average value  $I_0$ . The distribution is strongly asymmetric, with the Gaussian tale at  $I \gg I_0$ . Inset: The electron occupation  $f$  inside the wire as a function of the rescaled coordinate  $z$ , under the condition that the average current  $I = I_0$ , no current  $I = 0$ , and large current  $I = 5I_0$  has been measured.

expressing  $\mathcal{H}$  and  $\chi$  in terms of  $\mathcal{I}$ , we may also formally obtain the logarithm of the current distribution,

$$\ln P(I) = S(\mathcal{I}) - t\mathcal{I}\chi(\mathcal{I}), \quad \mathcal{I} \rightarrow I, \quad (21)$$

as a result of the stationary phase approximation for the integral  $P(I) = \int d\chi \exp[S(\chi) - t\mathcal{I}\chi]$  and because  $\partial\mathcal{H}/\partial\chi = \mathcal{I}/L$ .

As an example of the 1D field theory, we consider the FCS of the electron charge transmitted through the mesoscopic diffusive wire. When the potential difference  $\Delta\mu = \mu_L - \mu_R > 0$  is applied to the wire, the electrons flow from the left lead to the right lead with the average current  $I_0 = e^{-1}G\Delta\mu$ , where  $G$  is the conductance of the wire. The elastic electron scattering causes non-equilibrium fluctuations of the current. At zero temperature, and for noninteracting electrons (the cold electron regime), the FCS of the transmitted charge has been studied in Refs. [14] and [4] using quantum-mechanical methods with the following result for the generating function of cumulants of the dimensionless charge  $Q/e$ :

$$S(\chi) = (tI_0/e) \operatorname{arcsinh}^2 \left[ \sqrt{\exp(\chi) - 1} \right]. \quad (22)$$

Here we will rederive this result using our classical method.

On the classical level, the electrons in the diffusive wire are described by the distribution function  $f(z)$ . Under transport conditions (and at zero temperature), this distribution  $f(z)$  varies from  $f_L = 1$  in the left lead to  $f_R = 0$  in the right lead. Taking the continuum limit for the series of mesoscopic cavities,<sup>11</sup> we arrive at the action (16) in the form

$$S = (tI_0/e) \int_{-1/2}^{1/2} dz [-f'\lambda' + f(1-f)(\lambda')^2], \quad (23)$$

where we have rescaled the coordinate  $z$ ,  $\rho(z)$  has been replaced with the distribution  $f(z)$ , and where  $D = 1$ , and  $F = 2f(1-f)$  up to the overall constant  $I_0/e$ . This form of  $F$ , originating from the Pauli blocking factors, is quite general for fermionic systems. Applying now the procedure described in the beginning of this section, we solve the saddle-point equations and find the fields

$f$  and  $\lambda$ ,

$$f(z, \chi) = \frac{1}{2} \left[ 1 - \frac{\sinh(2\alpha z)}{\sinh \alpha} \right], \quad (24)$$

$$\lambda(z, \chi) = 2 \operatorname{arctanh} [\tanh(\alpha/2) \tanh(\alpha z)] , \quad (25)$$

$$\alpha = \operatorname{arcsinh} \left[ \sqrt{\exp(\chi) - 1} \right], \quad (26)$$

where  $\mathcal{H} = \alpha^2$ , so that according to the Eq. (20) we immediately obtain the result (22).

The logarithm of the current distribution  $\ln[P(I)]$  can be now found from the equation (21). We obtain the following result:

$$\ln[P(I)] = -(tI_0/e)[2\alpha \coth \alpha \ln(\cosh \alpha) - \alpha^2], \quad (27)$$

where  $\alpha$  has to be expressed in terms of  $\mathcal{I} = I/I_0$  by solving the equation

$$\alpha \coth \alpha = I/I_0. \quad (28)$$

The distribution  $P(I)$  is strongly asymmetric around the average current  $I = I_0$  (see Fig. 2).

In Ref. [1] we have proven the universality of the FCS of the transmitted charge for a two-terminal multi-dimensional generalized wire with the noise tensor  $F(\rho)\hat{T}$ , being an arbitrary function of the charge density  $\rho$ , and with the constant diffusion tensor  $D\hat{T}$ . The universality means that the FCS depends neither on the shape of the conductor, nor on its dimensionality. The FCS of a mesoscopic wire given by Eq. (22) is a particular example of universal FCS. In the more general case, when  $D$  is a function of  $\rho$ , the FCS depends on the geometry through only one parameter, the geometrical conductance.

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